The Geometry of Optical Paths*

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Outline

Motivation

What are optical paths?

What types of questions do I intend to pose and (maybe) answer?

Initial observations

Theorems about isometry

Existence (or non-existence) of extremal paths

Conjectures and open questions

Measure-theoretic problems

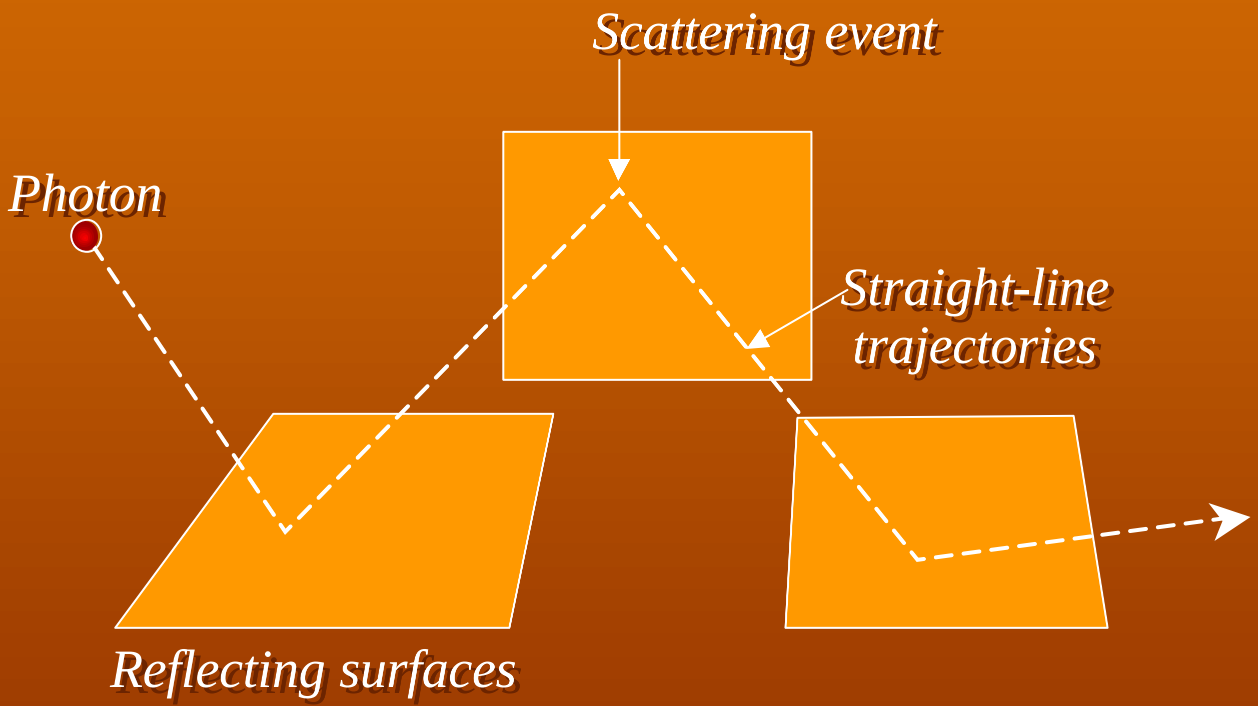
Concentration of measure

Potential applications

Error bounded radiative transfer

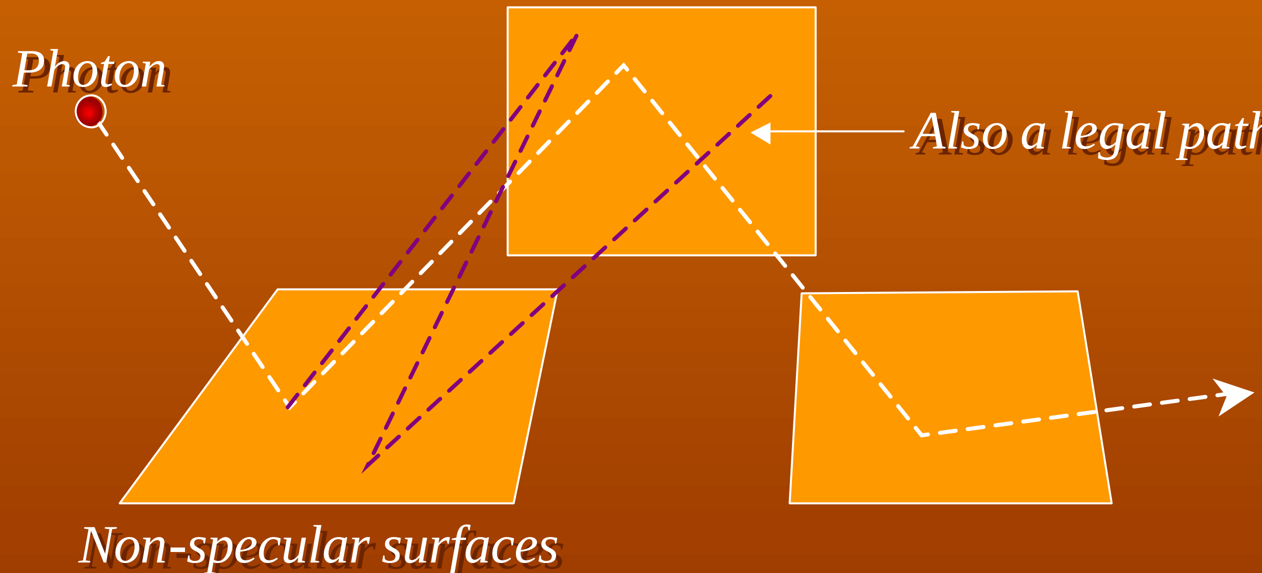
Efficient sampling of path space

Optical paths

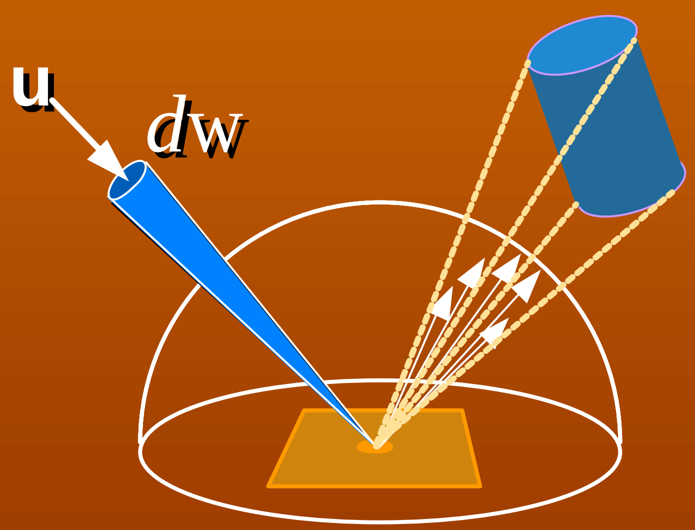


Optical path = “any trajectory a photon can follow”

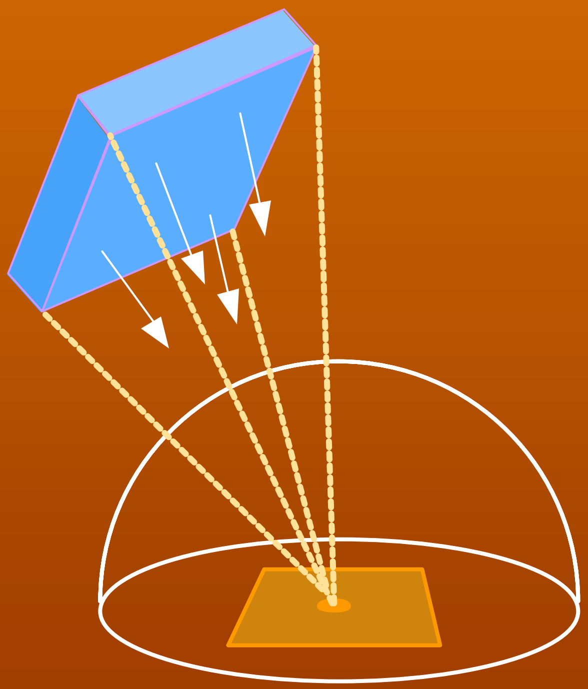
General optical paths



Integration over optical paths

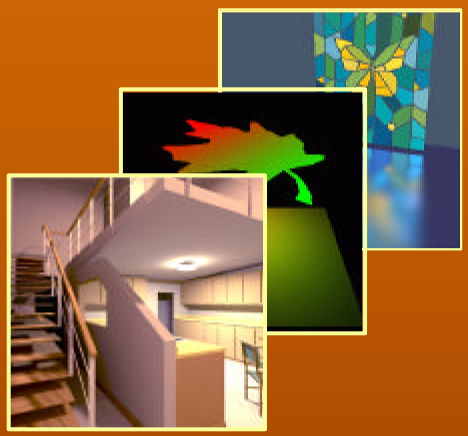


Integrate differential radiance reaching a second surface.

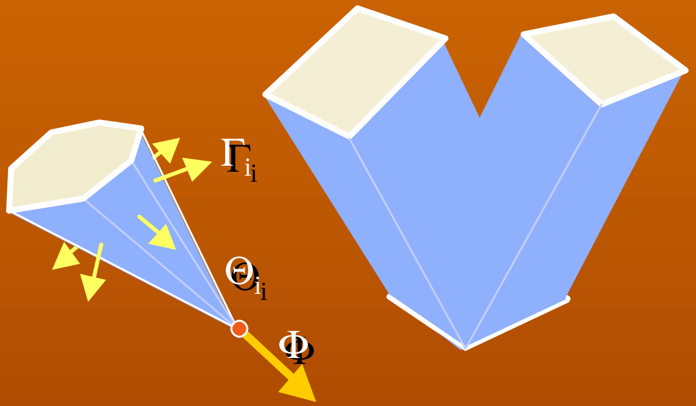


Integrate radiance over projected solid angle to get *irradiance*.

Motivation for this work

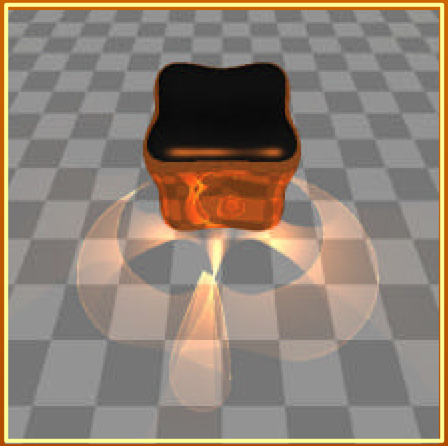


An image is a collection of “measurements”.



Measurements imply integration.

Motivation for this work



Mitchell & Hanrahan

Compute extremal paths.

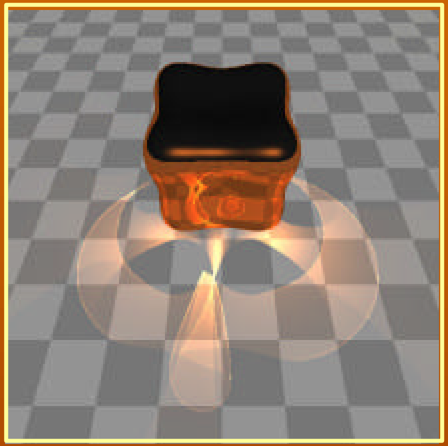
Smits & Arvo

Truncate or approximate long paths.

Veach & Guibas

Find representatives for each class of paths.

Motivation for this work



Mitchell & Hanrahan

Smits & Arvo

Veach & Guibas

What fundamental "geometrical" properties do the set of paths in an environment possess and how are reflectance and geometry entwined?

Types of questions to ask

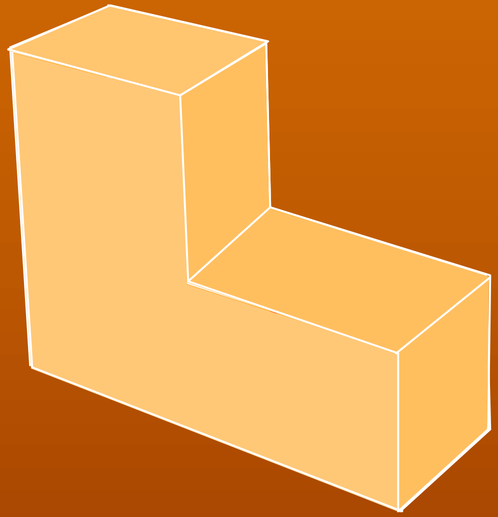
What are the appropriate measures on paths?

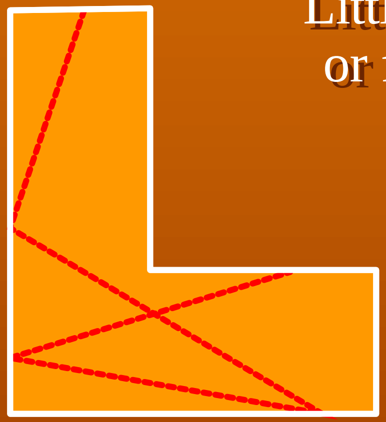
Are there extremal paths in general?

What can the family of paths tell us about the environment?

Do non-trivial reflectance functions add a new spin to old problems?

Paths in 3D and 2D





Little changes from 3D to 2D, or from curved to polygonal surfaces.

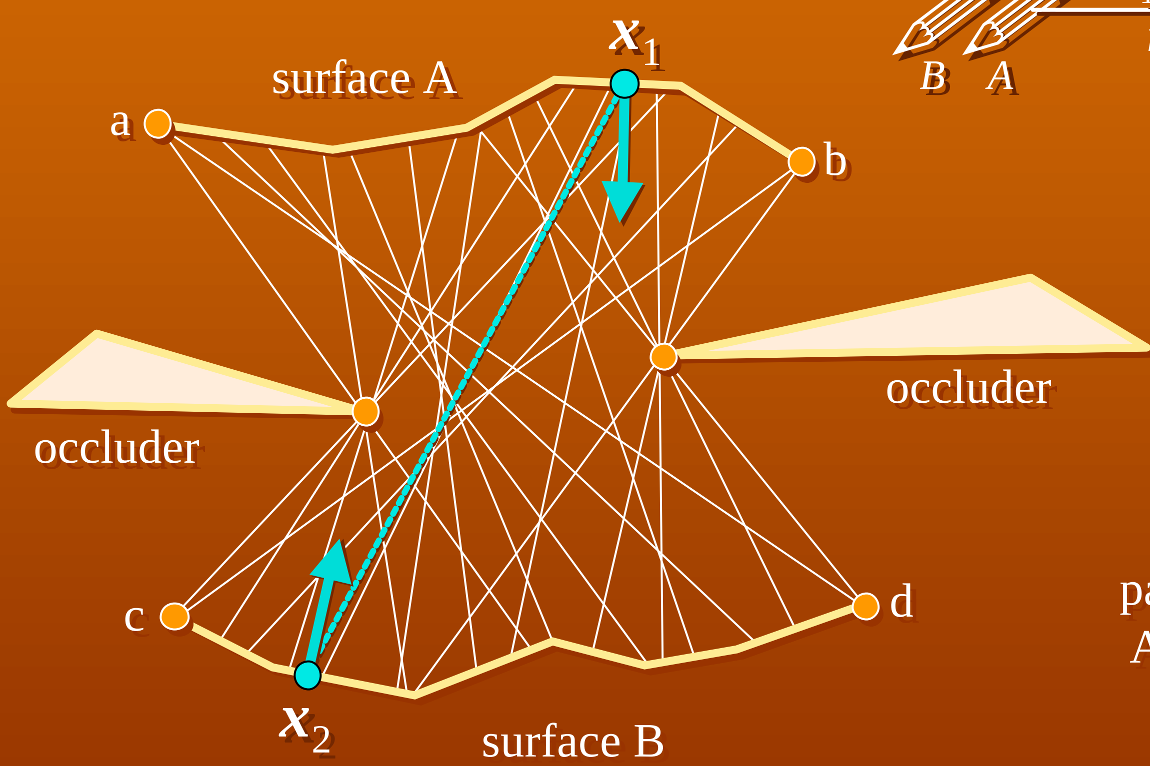
$$\frac{\cos\theta_1 \cos\theta_2}{r^2}$$

$$\frac{\cos\theta_1 \cos\theta_2}{r}$$

This is one change

Hottel's method

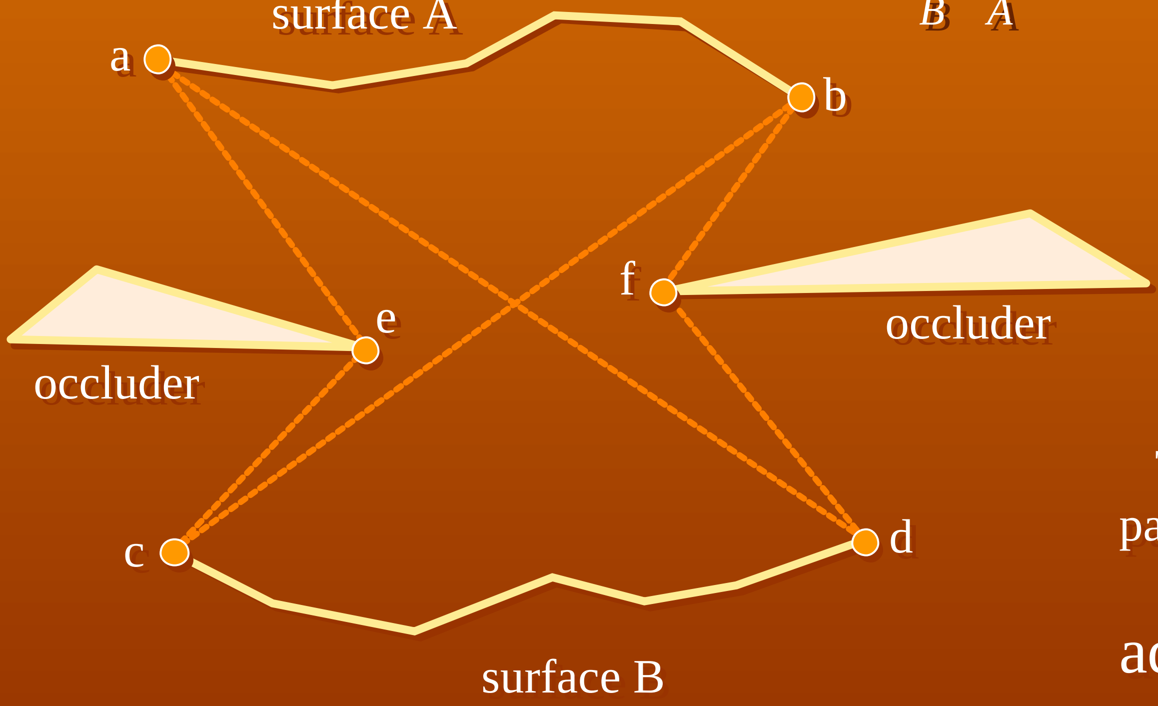
$$_{B}\ _{A}\ \frac{\cos\theta_1 \cos\theta_2}{r}\, v(x_1, x_2)\, dx_1\, dx_2$$



The measure of lines passing through surfaces A and B is proportional to the *form factor*.

Hottel's method

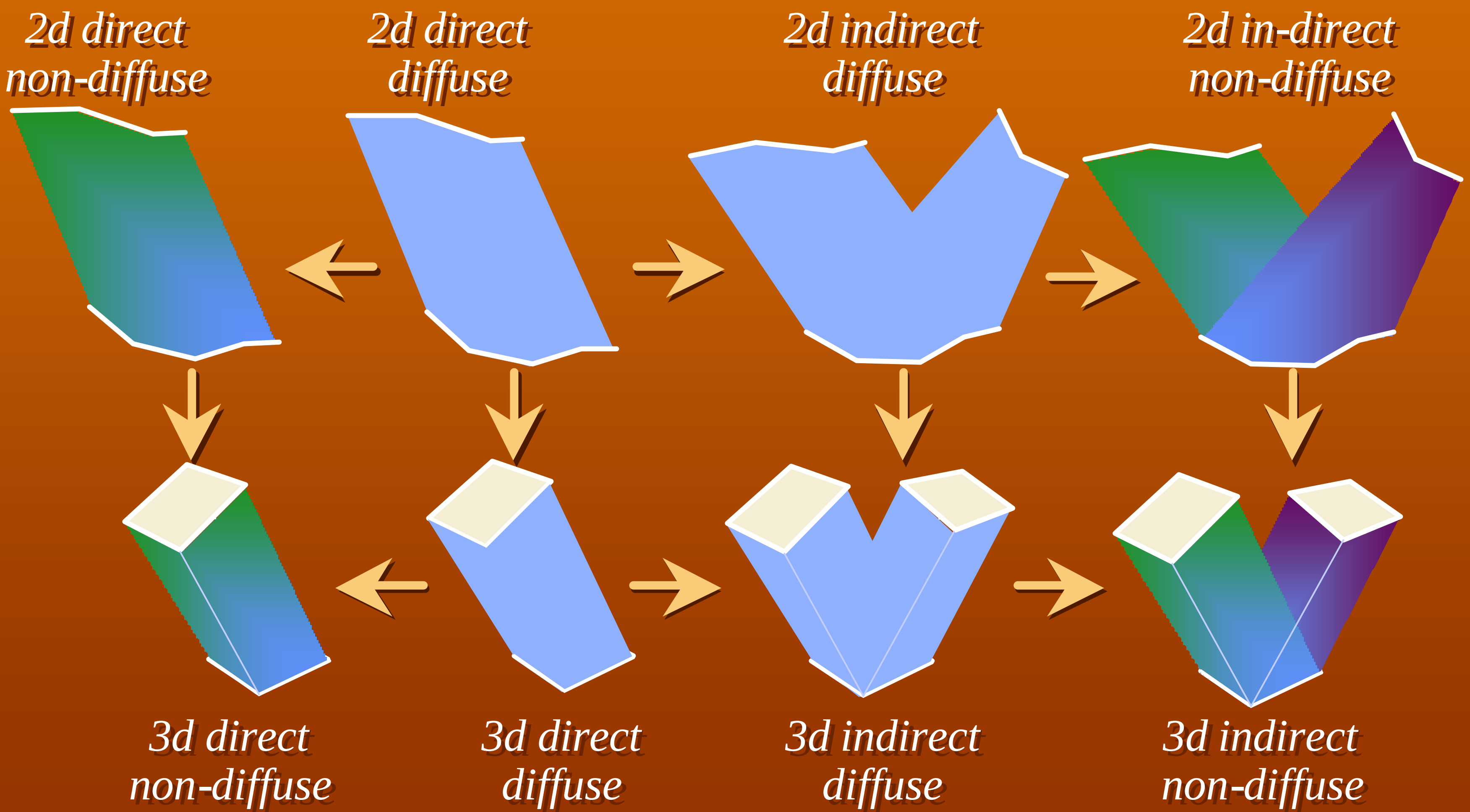
$$\int_B \int_A \frac{\cos\theta_1 \cos\theta_2}{r}\, v(x_1, x_2)\, dx_1\, dx_2$$



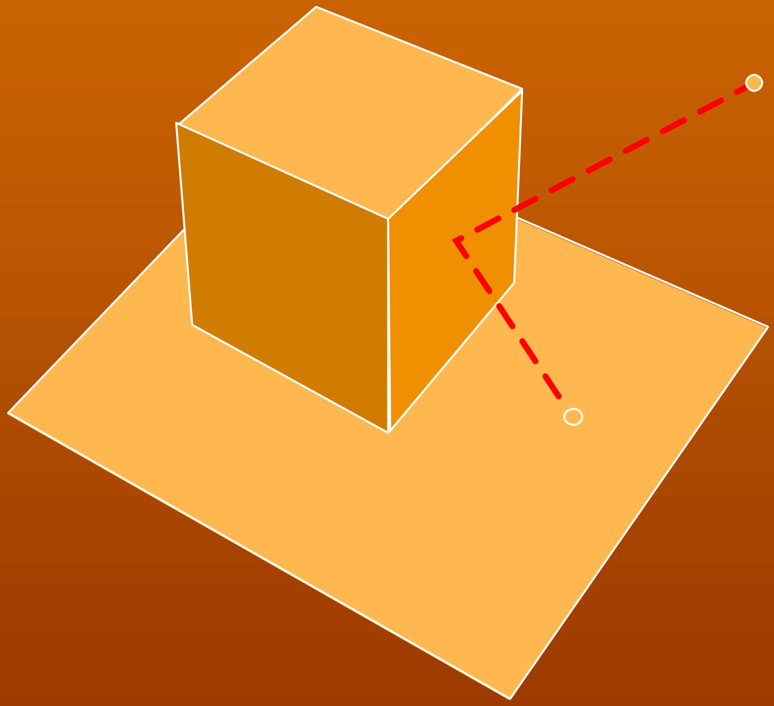
The measure of lines passing through surfaces A and B is

ad + bc – aec – bfd

Possible analogs of Hottel's method



Extremal paths

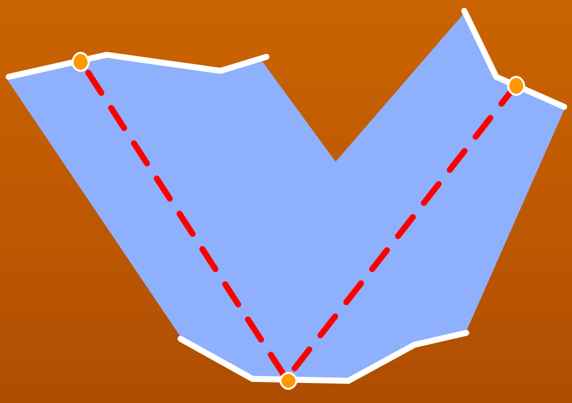


Extremal paths with respect to length (time).

Extremal paths with respect to gaussian curvature.

Is there an analogy for general reflection?

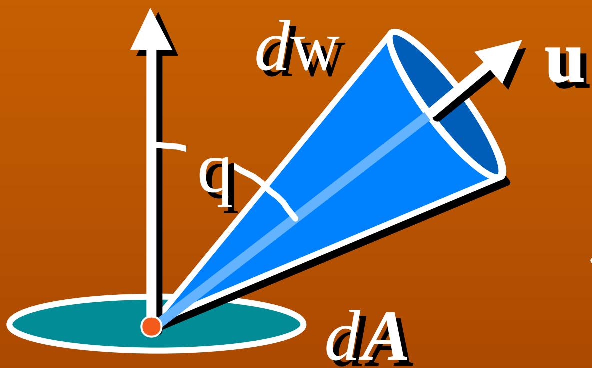
Extremal paths



What does extremal mean in this context?

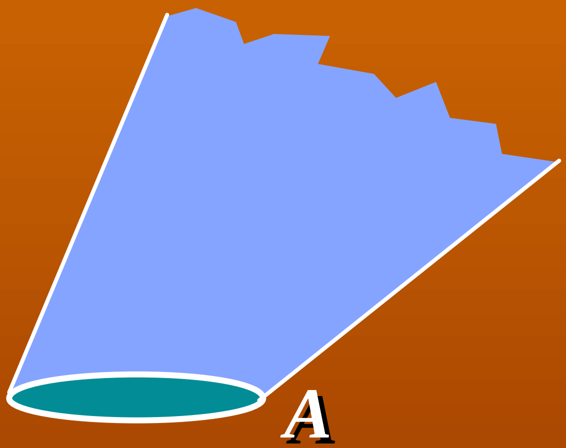
Radiance as a density and derivative

Classical view



$$\frac{\text{watts}}{\text{m}^2\ \text{sr}}$$

Measure-theoretic view

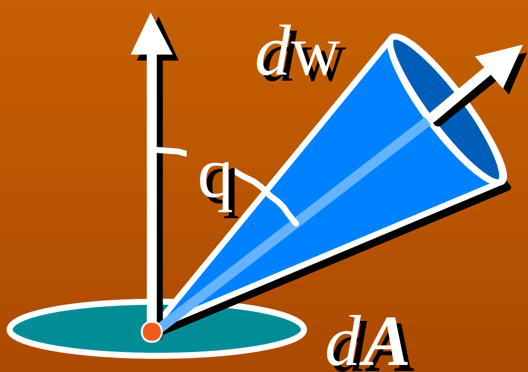


ϕ ⑥ σ ⑨ *m*

power measure — solid angle measure — area measure

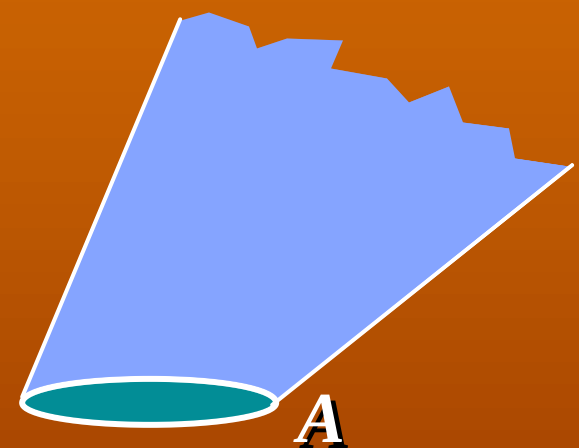
Radiance as a density and derivative

Classical view



$$\frac{\text{watts}}{\text{m}^2\ \text{sr}}$$

Measure-theoretic view

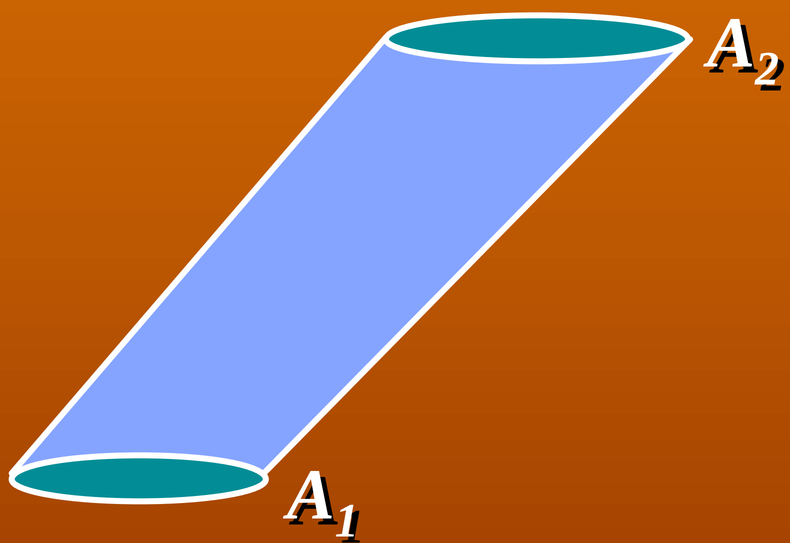


φ ⑥ σ ⑨ *m*

$$L = \frac{d\,\phi}{d\,\sigma \text{ ⑨ } m}$$

Radon-Nikodym derivative

Transport intensity as a derivative

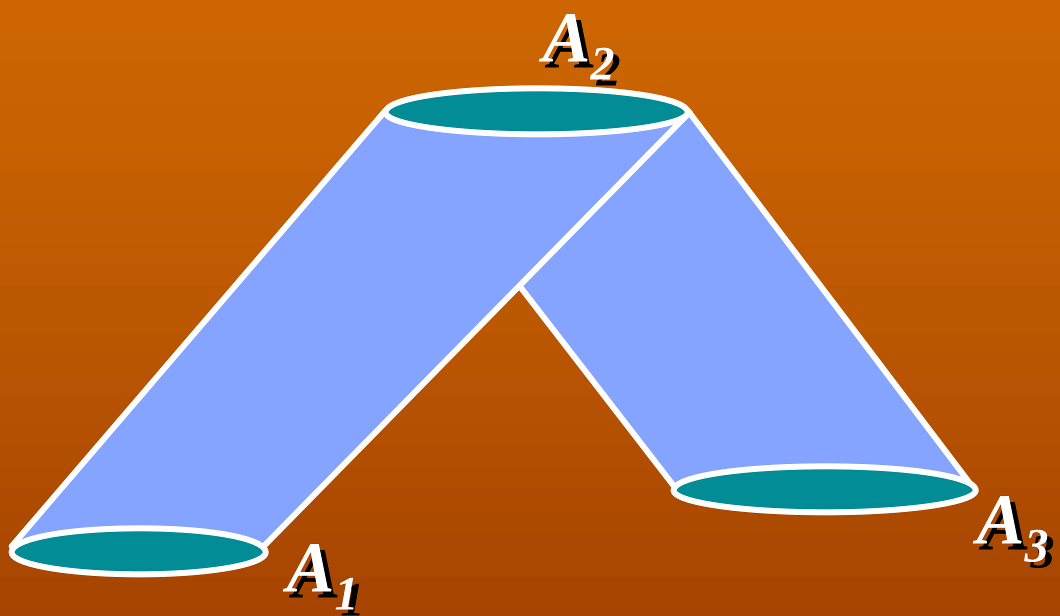


ϕ ⑥ m ⑨ m

$$I = \frac{d\,\phi}{d\,m \text{ ⑨ } m}$$

Radon-Nikodym derivative

Transport intensity as a derivative

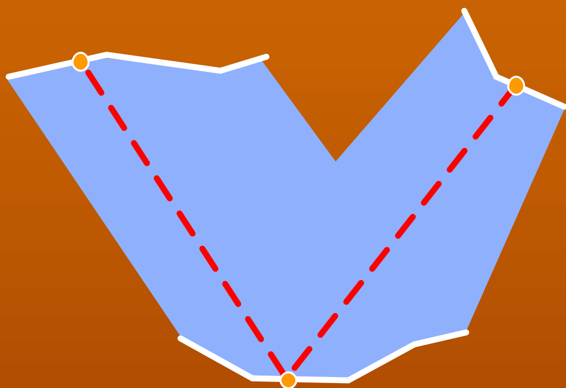


ϕ ⑥ *m* ⑨ *m* ⑨ *m*

I = d ϕ / d *m* ⑨ *m* ⑨ *m*

Radon-Nikodym derivative

Extremal paths

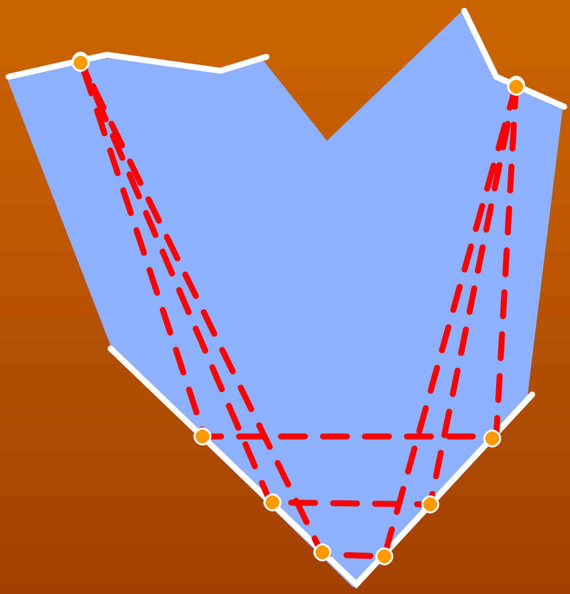


What does extremal mean in this context?

Extremal in $$\frac{d\,\phi}{d\,m \text{ ⑨ } m \cdots m}$$

or $$\text{⌫} \frac{\boldsymbol{cos}\,\theta_1\ \boldsymbol{cos}\,\theta_2}{\boldsymbol{r}}$$

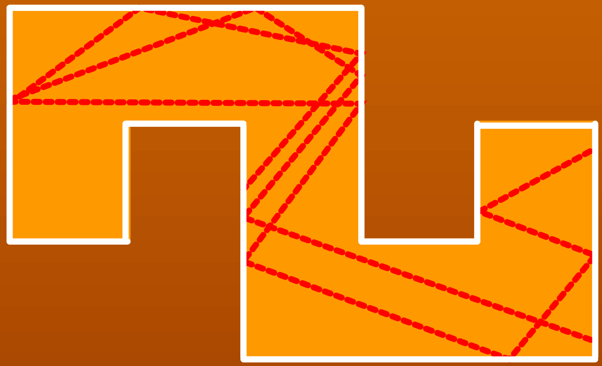
Extremal paths



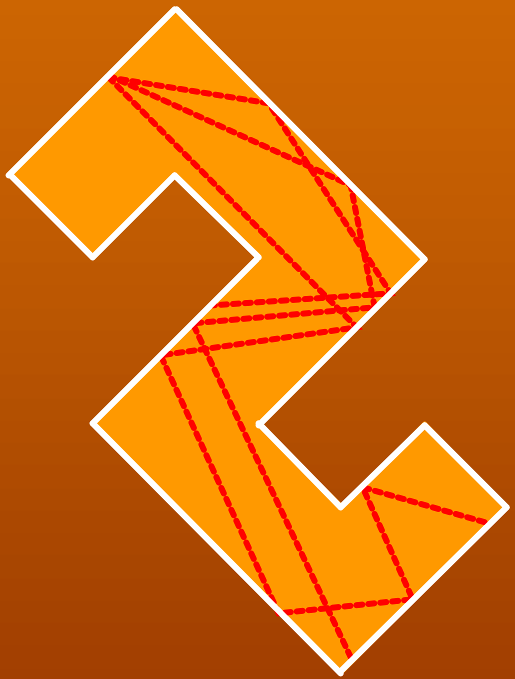
But the extremum does not always exist!

$$\frac{\cos\theta_1 \cos\theta_2}{r}$$

Path isometry

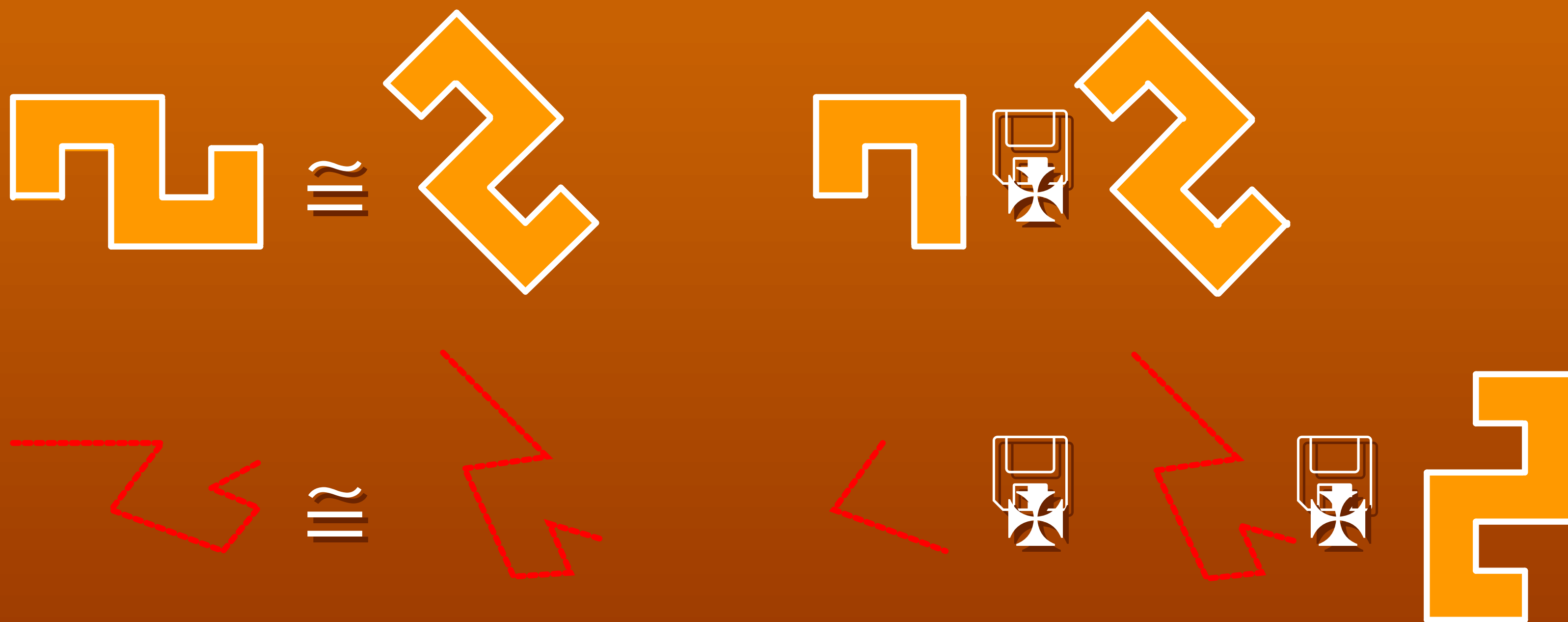


This collection of paths…

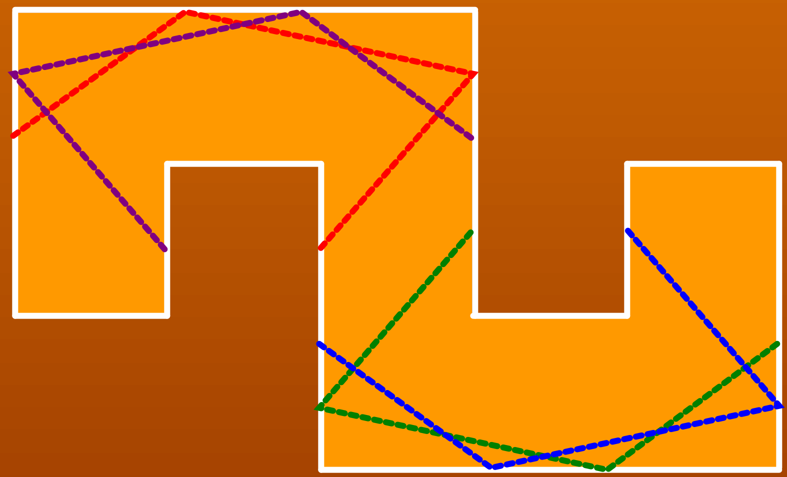


Is fundamentally the same as this collection of paths.

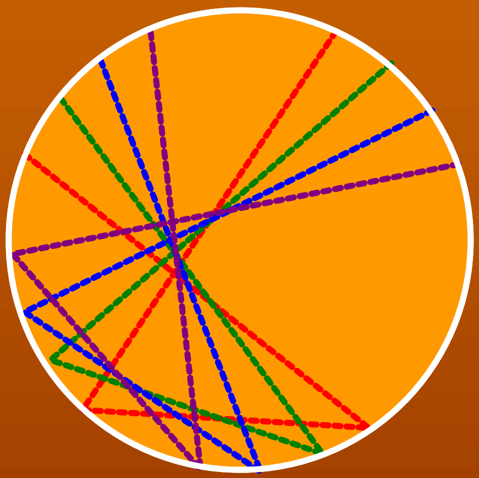
Path isometry



Path isometry



Finite number of isometric
paths within the same enclosure



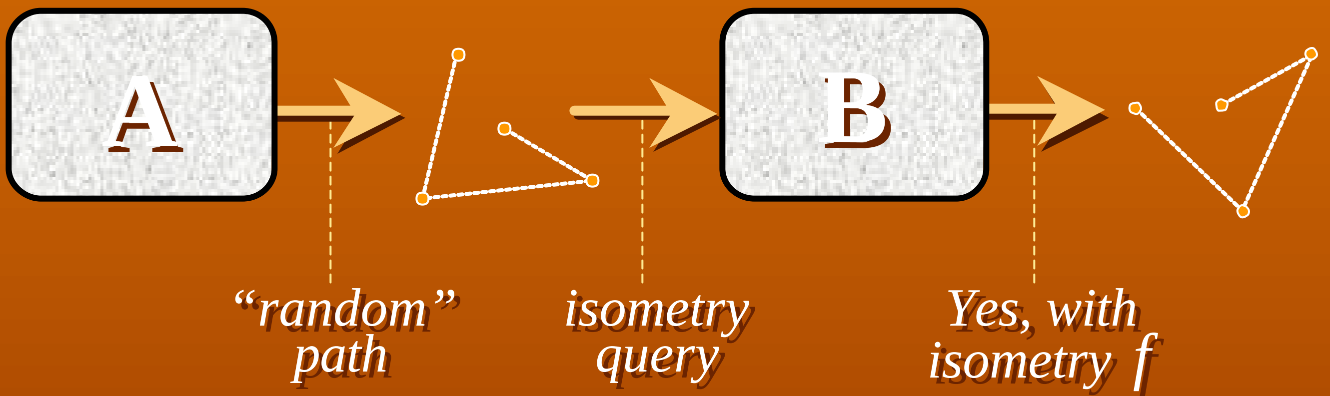
Continuum of isometric
paths within the same enclosure

Path isometry

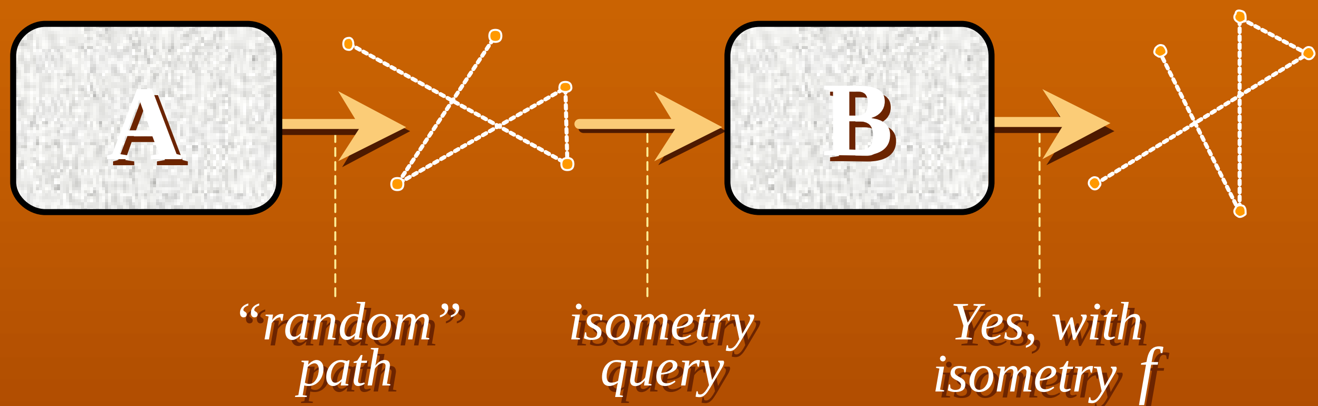
A

B

Path isometry

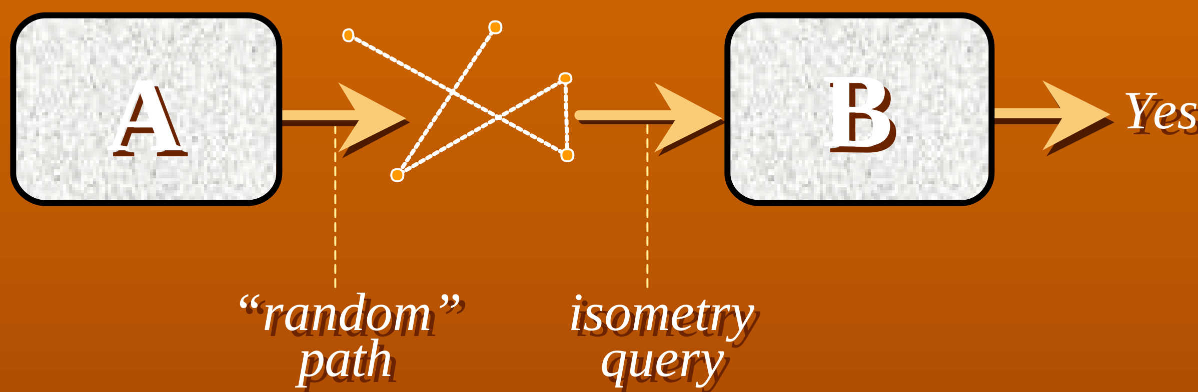


Path isometry



☎∀p♑A f B✆ f A B

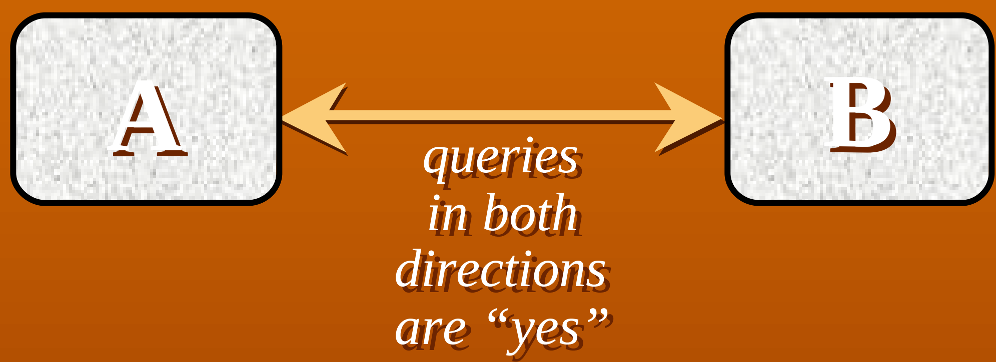
Path isometry



☎∀p♑A🖫 B✆ 🖫📂 A B

irrespective of the isometry

Path isometry



$$A \sqsubseteq B \wedge B \sqsubseteq A \Leftrightarrow A \cong B$$

irrespective of the isometry

Sub-isometry theorem

If A and B are *compact* sets, then

☎ A 🖫✠ B ⮝ B 🖫✠ A ✆ 🗁 $A \cong B$

If A and B are sets, then

$|A| \leq |B|$ ⮝ $|B| \leq |A|$ 🗁 $|A| = |B|$

(Schröder-Bernstein theorem)

Sub-isometry theorem

If A and B are ~~*compact*~~ sets, then

A B B A $A \cong B$

Concluding thoughts

Exact isometry cannot be determined in this way, but it may provide insight into equivalence classes of environments.

Correct definition/interpretation of extremal paths may lead to useful analogs of Hottel's method.

Measures on paths lead to new versions of classical problems in computational geometry.